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D=11 supergravity with manifest supersymmetry

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Abstract: The complete supersymmetric action for eleven-dimensional supergravity is presented. The action is polynomial in the scalar fermionic pure spinor superfield, and contains only a minor modification to the recently proposed three-point coupling.

1. INTRODUCTION

Eleven-dimensional supergravity [1] is the low-energy limit of the (not yet defined) M-theory, and hence of a strong-coupling limit of string theory. Having maximal supersymmetry, a traditional superspace description [2] puts the theory on-shell. Recently, a programme was initiated to formulate eleven-dimensional supergravity with manifest supersymmetry using a pure spinor superfield. A simple three-point interaction was proposed [3].

Pure spinor superfields provide a powerful tool for formulating supersymmetric field and string theories [4-18]. In models with maximal supersymmetry, the constraint on the ordinary superfield, which enforces the equations of motion, is encoded in a cohomological equation of the type $Q\Psi + \dots = 0$, which is the equation of motion for the pure spinor superfield. Pure spinor superfield theory inevitably leads to a Batalin-Vilkovisky (BV) formalism [19,20].

In the present paper, we will show that the deformation of the free action represented by the three-point interaction of ref. [3] is almost the whole answer. Due to the simple properties of the operators involved, higher order interactions are essentially absent. Pure spinor superfield formulations tend to have some remarkable properties, as an extra bonus in addition to the manifest supersymmetry. The action for $D = 10$ super-Yang-Mills is Chern-Simons-like, and has only a cubic interaction [8]. The conformal models in $D = 3$, whose component actions contain couplings of six scalar fields, simplify enormously in the pure spinor framework, where the matter superfields only have a minimal coupling to the Chern-Simons field [17,18]. Higher order interactions arise when auxiliary fields are eliminated (in both cases the fermionic component of the gauge connection on superspace). This type of simplification is shared by $D = 11$ supergravity, surprisingly to the extent that the action becomes polynomial.

The organisation of the paper is as follows. In Section 2, we review the construction of ref. [3]. The full action is given in Section 3, where we also expand the action around a background. Section 4 contains conclusions and a discussion, where we focus on identifying future directions of research.

2. ELEVEN-DIMENSIONAL SUPERGRAVITY WITH PURE SPINORS

The relevant pure spinors in $D = 11$ satisfy $(\lambda\gamma^a\lambda) = 0$. It has been known for some time that the cohomology of a scalar fermionic superfield under the the BRST operator $q = (\lambda D)$ gives the linearised supergravity multiplet [9,12,21]. The fermionic derivatives anticommute to give torsion $\{D_\alpha, D_\beta\} = -T_{\alpha\beta}{}^c D_c$, with $T_{\alpha\beta}{}^c = -2\gamma_{\alpha\beta}^c$. This holds for any background satisfying the equations of motion, but for all purposes in this paper D_α will be the flat covariant derivative. The fermionic scalar field Ψ has dimension -3 and ghost number 3,

and its lowest component is the third order ghost for the tensor field. The physical fields of ghost number 0 sit in the field as $\lambda^\alpha \lambda^\beta \lambda^\gamma C_{\alpha\beta\gamma}(x, \theta)$, where λ^α is the pure spinor and $C_{\alpha\beta\gamma}$ the lowest-dimensional part of the superspace 3-form C . There is a natural measure on the pure spinor space, and it is straightforward to write an action $\int \Psi Q \Psi$ giving the linearised equations of motion [21]. The integrand has ghost number 7 and dimension -6 . We refer to ref. [3] for details and conventions.

In refs. [9,12] it was shown that there is also another field, Φ^a , of dimension -1 and ghost number 1, that contains the linearised multiplet. This field has the additional gauge symmetry $\Phi^a \approx \Phi^a + (\lambda \gamma^a \rho)$, and has as its lowest component the diffeomorphism ghost. The physical fields sit in $\lambda^\alpha h_\alpha^a$, where h_α^a is the linearised supervielbein.

It is necessary, both for having a non-degenerate measure and in order to write the relevant operators, to work with non-minimal pure spinors [15]. In addition to the pure spinor λ , one has the pure spinor $\bar{\lambda}$ and the fermionic spinor r which is pure relative to $\bar{\lambda}$, $(\bar{\lambda} \gamma^a r) = 0$. The non-minimal BRST operator is $Q = q + s = (\lambda D) + (r \frac{\partial}{\partial \lambda})$.

2.1. THE THREE-POINT COUPLING

In our recent paper [3], we constructed a three-point coupling for eleven-dimensional supergravity. This was done by constructing the BRST-invariant operator R^a relating the two fields according to $\Phi^a = R^a \Psi$. It takes the form

$$\begin{aligned} R^a &= R_0^a + R_1^a + R_2^a \\ &= \eta^{-1} (\bar{\lambda} \gamma^{ab} \bar{\lambda}) \partial_b + \eta^{-2} (\bar{\lambda} \gamma^{ab} \bar{\lambda}) (\bar{\lambda} \gamma^{cd} r) (\lambda \gamma_{bcd} D) \\ &\quad - 16 \eta^{-3} (\bar{\lambda} \gamma^{a[b} \bar{\lambda}) (\bar{\lambda} \gamma^{cd} r) (\bar{\lambda} \gamma^{e]f} r) (\lambda \gamma_{fb} \lambda) (\lambda \gamma_{cde} w) . \end{aligned} \quad (2.1)$$

Some alternative ways of writing the last term are given in Appendix A. Here, the invariant η is defined as $\eta = (\lambda \gamma^{ab} \lambda) (\bar{\lambda} \gamma_{ab} \bar{\lambda})$

The action of ref. [3] is

$$S_3 = \int [dZ] \left[\frac{1}{2} \Psi Q \Psi + \frac{1}{6} (\lambda \gamma_{ab} \lambda) \Psi R^a \Psi R^b \Psi \right] . \quad (2.2)$$

In order to show that the BV master equation is fulfilled to third order in the field, we only need to use

$$(\lambda \gamma_{ab} \lambda) [Q, R^b] = 0 , \quad (2.3)$$

which was how R^a was constructed in ref. [3]. (There, R^a was viewed as an operator from the space of scalar functions Ψ to the space of vectorial functions Φ^a with the extra gauge invariance $\Phi^a \approx \Phi^a + (\lambda\gamma^a \varrho)$. The factor $(\lambda\gamma_{ab}\lambda)$ in eq. (2.3) encodes this invariance.)

2.2. AN EXAMPLE: THE CHERN–SIMONS TERM

In ref. [3], it was shown that one of the terms contained in the three-point coupling gave the ghost couplings appropriate for the diffeomorphism algebra. We would like to complement that example with one clearly displaying how a known interaction among physical supergravity fields, namely the supergravity Chern–Simons term $\int C \wedge H \wedge H$, is generated. The cohomology for the C -field is [22]

$$\Psi_C \sim (\lambda\gamma^i\theta)(\lambda\gamma^j\theta)(\lambda\gamma^k\theta)C_{ijk} + \dots, \quad (2.4)$$

where the ellipsis denotes terms with $H = dC$ (which are higher order in θ). Acting with R_0^a gives $R_0^a\Psi_C \sim \eta^{-1}(\bar{\lambda}\gamma^{ai}\bar{\lambda})(\lambda\gamma^j\theta)(\lambda\gamma^k\theta)(\lambda\gamma^l\theta)\partial_i C_{jkl} + \dots$. The integrand in the coupling term becomes

$$\begin{aligned} &\sim \eta^{-1}(\bar{\lambda}\gamma^{ij}\bar{\lambda})(\lambda\gamma^k\theta)(\lambda\gamma^l\theta)(\lambda\gamma^m\theta)C_{klm} \\ &\quad \times (\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\lambda\gamma^q\theta)\partial_i C_{npq}(\lambda\gamma^r\theta)(\lambda\gamma^s\theta)(\lambda\gamma^t\theta)\partial_j C_{rst}. \end{aligned} \quad (2.5)$$

We now use the identity [21] $(\lambda\gamma^{i_1}\theta) \dots (\lambda\gamma^{i_9}\theta) \sim \varepsilon^{i_1 \dots i_9 ab}(\lambda\gamma_{ab}\lambda)\mathcal{N}$, where \mathcal{N} is the scalar cohomology at $\lambda^7\theta^9$ used in the measure. We also replace $\partial_i C_{jkl}$ by H_{ijkl} in eq. (2.5), since the only way to form a scalar is “ $C \wedge H \wedge H$ ”. Inserting this in (2.5) directly gives $\sim \mathcal{N}\varepsilon^{i_1 \dots i_{11}}C_{i_1 i_2 i_3}H_{i_4 i_5 i_6 i_7}H_{i_8 i_9 i_{10} i_{11}}$, without the need of adding any q -exact terms. This shows that the supergravity Chern–Simons term, and hence by supersymmetry all supergravity 3-point couplings, are contained in the interaction term.

3. THE COMPLETE DYNAMICS

3.1. THE FULL ACTION

We will now examine the master equation to higher order. The master equation reads

$$(S, S) = 0, \quad (3.1)$$

where the antibracket is defined as

$$(A, B) = \int A \frac{\overleftarrow{\delta}}{\delta \Psi(Z)} [dZ] \frac{\overrightarrow{\delta}}{\delta \Psi(Z)} B . \quad (3.2)$$

We begin by performing a variation of the action (2.2):

$$\begin{aligned} \delta S_3 &= \int [dZ] \delta \Psi \left[Q \Psi + \frac{1}{6} (\lambda \gamma_{ab} \lambda) R^a \Psi R^b \Psi + \frac{1}{3} R^a ((\lambda \gamma_{ab} \lambda) \Psi R^b \Psi) \right] \\ &= \int [dZ] \delta \Psi \left[Q \Psi + \frac{1}{2} (\lambda \gamma_{ab} \lambda) R^a \Psi R^b \Psi + \frac{1}{3} \Psi R^a ((\lambda \gamma_{ab} \lambda) R^b \Psi) \right] . \end{aligned} \quad (3.3)$$

The remainder from the antibracket is

$$(S_3, S_3) = \frac{1}{3} \int [dZ] (\lambda \gamma_{ab} \lambda) R^a \Psi R^b \Psi \Psi R^c ((\lambda \gamma_{cd} \lambda) R^d \Psi) . \quad (3.4)$$

Here, we have already used $(\lambda \gamma_{ab} \lambda) (\lambda \gamma_{cd} \lambda) R^a \Psi R^b \Psi R^c \Psi R^d \Psi = 0$, which follows from the pure spinor constraint. If the last term would vanish, *i.e.*, if $R^a (\lambda \gamma_{ab} \lambda) R^b = 0$, the action S_3 would be the full action. We will now show that this is not true, but almost so, in the sense that $R^a ((\lambda \gamma_{ab} \lambda) R^b \Psi)$ is a cohomologically trivial field.

The detailed calculation is performed in Appendix B. It is a bit lengthy, but once it is performed it leads to very simple properties for the operators. The calculation in Appendix B shows that

$$R^a (\lambda \gamma_{ab} \lambda) R^b = \frac{1}{2} (\lambda \gamma_{ab} \lambda) [R^a, R^b] = \frac{3}{2} \{Q, T\} . \quad (3.5)$$

where we have defined the fermionic operator T with dimension 3 and ghost number -3 as

$$T = 8\eta^{-3} (\bar{\lambda} \gamma^{ab} \bar{\lambda}) (\bar{\lambda} r) (r r) N_{ab} . \quad (3.6)$$

Note that $T\Psi$ is bosonic and has dimension as well as ghost number zero. It seems likely that the field $T\Psi$ is connected to the trace of the metric fluctuation, and thus to the determinant of the metric. So, the operator $R^a (\lambda \gamma_{ab} \lambda) R^b$ is zero in the cohomology. In addition, the operator T has very nice properties. Since it contains a multiplicative factor $(\bar{\lambda} r)$, it squares to zero, even when two T 's act on different fields, $TATB = 0$. Consequently, $TA\{Q, T\}B + \{Q, T\}ATB = 0$ (for fermionic A), and in particular $T\Psi\{Q, T\}\Psi = 0$. T commutes with R^a , and of course with the regularisation factor in the measure (since R^a does). It does not commute with $(\lambda \gamma_{ab} \lambda)$, but as long as there are contraction with R^a and R^b the commutator gives zero: $R^a A R^b B [T, (\lambda \gamma_{ab} \lambda)] C = 0$.

The remaining term in the master equations may now be written

$$(S_3, S_3) = \frac{1}{2} \int [dZ] (\lambda \gamma_{ab} \lambda) \Psi \{Q, T\} \Psi R^a \Psi R^b \Psi . \quad (3.7)$$

This term is cancelled by the antibracket between a term $-\frac{1}{4} \int [dZ] (\lambda \gamma_{ab} \lambda) \Psi T \Psi R^a \Psi R^b \Psi$ and the kinetic term in the action:

$$S = \int [dZ] \left[\frac{1}{2} \Psi Q \Psi + \frac{1}{6} (\lambda \gamma_{ab} \lambda) (1 - \frac{3}{2} T \Psi) \Psi R^a \Psi R^b \Psi \right] . \quad (3.8)$$

With this slight modification of the action given in ref. [3], the master equation is exactly satisfied. Due to the simple algebraic properties of T , only terms to linear order in $T\Psi$ appear, and no new terms of higher order in Ψ are generated in (S, S) (see below for an explicit demonstration of this fact using the equation of motion). Somewhat surprisingly, we thus find that the full supergravity action in the pure spinor superfield formulation is polynomial.

The factor $1 - \frac{3}{2} T \Psi$ in the coupling term may be removed by performing a field redefinition $\tilde{\Psi} = (1 + \frac{1}{2} T \tilde{\Psi}) \tilde{\Psi}$ (which due to the nilpotency of T gives $T \tilde{\Psi} = T \Psi$ and the inversion $\tilde{\Psi} = (1 - \frac{1}{2} T \Psi) \Psi$). This leads to a non-canonical kinetic term:

$$\begin{aligned} S &= \int [dZ] \left[\frac{1}{2} (1 + T \tilde{\Psi}) \tilde{\Psi} Q \tilde{\Psi} + \frac{1}{6} (\lambda \gamma_{ab} \lambda) \tilde{\Psi} R^a \tilde{\Psi} R^b \tilde{\Psi} \right] \\ &= \int [dZ] \left[\frac{1}{2} e^{T \tilde{\Psi}} \tilde{\Psi} Q \tilde{\Psi} + \frac{1}{6} (\lambda \gamma_{ab} \lambda) \tilde{\Psi} R^a \tilde{\Psi} R^b \tilde{\Psi} \right] . \end{aligned} \quad (3.9)$$

This action is probably closer related to the geometric formulation than the action (3.8). Note that the field redefinition is not canonical with respect to the antibracket, which can be calculated using $\frac{\delta}{\delta \Psi} = (1 - T \tilde{\Psi}) \frac{\delta}{\delta \tilde{\Psi}} + \frac{1}{2} \tilde{\Psi} T \frac{\delta}{\delta \tilde{\Psi}}$.

The equation of motion following from the redefined action enjoys cancellations between terms from the kinetic term and the coupling term and reads

$$(1 + \frac{3}{2} T \tilde{\Psi}) Q \tilde{\Psi} + \frac{1}{2} (\lambda \gamma_{ab} \lambda) R^a \tilde{\Psi} R^b \tilde{\Psi} = 0 , \quad (3.10)$$

while the equation of motion for the canonical field is

$$Q \Psi + \frac{1}{2} \Psi \{Q, T\} \Psi + \frac{1}{2} (\lambda \gamma_{ab} \lambda) (1 - 2 T \Psi) R^a \Psi R^b \Psi = 0 . \quad (3.11)$$

Using the equation of motion, it is easy to show explicitly that the master equation is indeed satisfied to all orders. The master equation is equivalent to the vanishing of the integral of the square of the equation of motion (for the canonical field). The square of each of the three terms in eq. (3.11) gives zero, and the cross terms are

$$(S, S) = \int [dZ] [Q\Psi\Psi\{Q, T\}\Psi + (\lambda\gamma_{ab}\lambda)Q\Psi(1 - 2T\Psi)R^a\Psi R^b\Psi + \frac{1}{2}(\lambda\gamma_{ab}\lambda)\Psi\{Q, T\}\Psi R^a\Psi R^b\Psi] \quad (3.12)$$

Using the algebraic properties of the operators above, it is straightforward to show that the terms both at third and fourth order in Ψ combine into total derivatives.

Since the operators involve negative powers of $\eta = (\lambda\gamma_{ab}\lambda)(\bar{\lambda}\gamma^{ab}\bar{\lambda})$, the singular properties at $\eta = 0$ must be checked. In ref. [3], it was shown that the number of negative powers of $(\lambda\gamma^{ab}\lambda)$ or $(\bar{\lambda}\gamma^{ab}\bar{\lambda})$ must be smaller than 12 for an integral to converge. Each R^a has at most 4 negative powers, while T has 5. The expression (3.8) needs regularisation in order to be well defined. This can probably be achieved using a method similar to that of ref. [23], but it is not obvious to what extent the algebraic properties of R^a and T will be preserved by such a regularisation. The form (3.9) of the action, on the other hand, is well defined without regularisation.

3.2. EXPANSION AROUND A BACKGROUND

Let Ψ_0 be a solution to the equation of motion (3.11) and let $\Psi = \Psi_0 + \psi$. We choose to expand the “canonical” action (3.8), since the field ψ is canonical in the sense that the antibracket is

$$(A, B) = \int A \frac{\overleftarrow{\delta}}{\delta\psi(Z)} [dZ] \frac{\overrightarrow{\delta}}{\delta\psi(Z)} B, \quad (3.13)$$

which allows us to compare the expanded and original actions directly. An expansion of the non-canonical action (3.9) requires letting $\tilde{\Psi} = \tilde{\Psi}_0 + (1 - \frac{1}{2}T\tilde{\Psi}_0)\tilde{\psi} - \tilde{\Psi}_0 T\tilde{\psi}$, but can also be obtained from rescaling of result below in terms of ψ . Expanding around the solution gives

$$S = S[\Psi_0] + \int [dZ] \left[\frac{1}{2}\psi Q'\psi + \frac{1}{6}(\lambda\gamma_{ab}\lambda)(1 - \frac{3}{2}T\psi)\psi R'^a\psi R'^b\psi \right], \quad (3.14)$$

where

$$\begin{aligned} Q' &= Q + 2Q\Psi_0 T + (1 - 2T\Psi_0) [(\lambda\gamma_{ab}\lambda)R^a\Psi_0 R^b + \frac{1}{2}\Psi_0\{Q, T\}] , \\ R'^a &= (1 - T\Psi_0)R^a - 2R^a\Psi_0 T . \end{aligned} \quad (3.15)$$

It may be checked directly that $Q'^2 = 0$ when Ψ_0 fulfills the equation of motion. The commutators $(\lambda\gamma_{ab}\lambda)[Q', R'^b]$ (being 0 for $\Psi_0 = 0$) and $(\lambda\gamma_{ab}\lambda)R'^a R'^b$ (equalling $\frac{3}{2}\{Q, T\}$)

for $\Psi_0 = 0$), seem to become more complicated, however. Obviously, the master equation will hold, but we have not checked this explicitly using the primed operators. The master equation implies relations *e.g.* $(\lambda\gamma_{ab}\lambda)[Q', R'^a]\psi R'^b\psi = 0$, which are implied by the previous ones (the ones in a flat background) but may be weaker in the sense that they need contractions with fields.

It is nice to verify that there is a kind of weak background invariance, in the sense that the action in any background is given by the same formal expression, given by eq. (3.14), with background dependent operators fulfilling the same relations independent of background (although weaker relations than the ones used in the flat background). Since the model contains gravity, it is natural that the BRST operator encodes information about the background geometry. We have not yet been able to examine the connection between the action in a background and the corresponding construction starting from a solution in superspace. There, one would construct the BRST operator as $\mathcal{Q} = \lambda^\alpha D_\alpha = \lambda^\alpha E_\alpha^M \partial_M$, E_A^M being the inverse supervielbein of the background. It is reasonable to expect a relation (equality?) between the operators \mathcal{Q} and Q' of eq. (3.15), and also between interaction terms. In order to establish such a relation, one should perform the analogous construction to the one in the present paper and ref. [3] but in a curved background superspace. The calculation of Appendix B relies on the algebra of flat superspace derivatives, and has to be revised in other backgrounds. We find it likely that the calculation will stay formally unchanged with the flat derivatives replaced by covariant derivatives in other backgrounds, but this remains to be seen.

The fact that the action is polynomial of course gives a small hope of finding a truly background independent formulation. The solution $\Psi = 0$ corresponds to flat space. In a background independent formulation the expectation value 0 for the field would be a non-geometric situation, and flat space would arise through an expectation value of the field.

4. CONCLUSIONS

Contrary to the expectations expressed in ref. [3], the interaction term derived there turned out to be almost the complete answer. The action for eleven-dimensional supergravity turns out to be polynomial, and only contains up to four-point couplings (or three-point, after a field redefinition). Once the algebraic relations between the operators used in the construction are derived, the construction encodes the full nonlinear structure of the supergravity in an extremely simple way. The efficiency of the pure spinor formalism in reducing the complexity of supersymmetric dynamics, already demonstrated for $D = 10$ super-Yang-Mills theory [8] and the BLG and ABJM models in $D = 3$ [17,18], turns out to be present also for supergravity. We have no clear understanding why this happens.

The formulation was made specifically in a flat background, although it was shown in Section 3.2 that the action takes the same formal expression in any background. We have however not yet been able to relate that action to one obtained from the supergeometry of the background. The geometric status of the supersymmetric action is somewhat unclear. It should be stressed, though, that the full gauge invariance, consisting of superdiffeomorphisms and tensor gauge symmetry (together with an infinite number of cohomologically trivial symmetries), is present, if not completely covariant. The precise relation of the pure spinor formulation and the geometric formulation needs to be clarified. Such a relation would hopefully resolve the issue of background invariance. Maybe some improvement of the pure spinor action could make it more geometric and simplify the comparison. One possibility may be the introduction of a field Ω^a_b containing a spin connection, making the Lorentz symmetry local. On the other hand, it seems to some extent to be the “de-geometrisation” of the action that allows for the supersymmetric formulation.

It may not be as strange as it sounds to have a polynomial action for gravity, once auxiliary fields are included. Recall the first order formulation of gravity with an independent spin connection, $S \sim \int \varepsilon_{a_1 \dots a_d} e^{a_1} \wedge \dots \wedge e^{a_{D-2}} \wedge R^{a_{D-1} a_D}(\omega)$. The equation of motion from varying the spin connection is the torsion-free condition on the vielbein, which eliminates the spin connection as an independent field. The dynamics becomes non-polynomial when the torsion constraint is solved, since the solution involves the inverse vielbein. Something similar may be happening in the pure spinor formalism. There is more than enough room in the superfield at ghost number 0 to accommodate a spin connection.

The formulation treats metric and tensor degrees of freedom in a democratic way. This may open for a simple proof of U-duality in dimensional reductions of the model. We envisage two possible ways of dealing with U-duality. One possibility is to try to incorporate the compact subgroup of the U-duality group as an enlarged structure group, which will involve new types of pure spinors and new cohomology. Another would be to try to realise U-duality operators on the field Ψ as ghost number 0 operators constructed with non-minimal pure spinors.

The quantum properties of $N = 8$ supergravity are not completely understood. A formulation with manifest supersymmetry would provide a good starting point. Some calculations have already been made for $D = 11$ supergravity using pure spinors in a superparticle formalism [24], but having a field-theoretic action from which amplitudes are derived will put the formalism on more solid ground. In order to use the action for calculating amplitudes one needs to perform gauge fixing. An essential part of this is to find the b -ghost, with the property $\{Q, b\} = \square$. The b -ghost for pure spinor superfields in $D = 10$ was given in ref. [15]. It is singular when $(\lambda\bar{\lambda}) = 0$, *i.e.*, at the tip of the pure spinor cone. The r -independent part of that operator, $b_0 \sim (\lambda\bar{\lambda})^{-1}(\bar{\lambda}\gamma^a D)\partial_a$ does not work in $D = 11$, in that it does not satisfy $\{s, b_0\} + \{q, b_1\} = 0$ for any b_1 . We envisage that the singular behaviour instead

comes with negative powers of $\eta = (\lambda\gamma_{ab}\lambda)(\bar{\lambda}\gamma^{ab}\bar{\lambda})$, like in the operator R^a . Work on gauge fixing is under way.

Gauge fixing of a BV action amounts to ordinary gauge fixing of the physical fields together with elimination of the antifields. A condition $b\Psi = 0$ is not the whole story, since one would like the antifields for the metric and tensor fields not to be set to zero, but to be related to the corresponding “antighosts” in a (field-theoretically) non-minimal BV formalism [20]. The standard procedures for gauge fixing (which treat fields and antifields asymmetrically) are not applicable in a setting where all fields and antifields reside in the single field Ψ . It is likely that extra fields need to be introduced, containing Nakanishi–Lautrup fields and antighosts. These aspects have to our knowledge not been addressed for pure spinor superfield theory, and should be investigated.

Finally, there will be need to regularise operators which diverge on some subspace of pure spinor space. We have not dealt with this problem yet, since at least one of the alternative forms of the action turned out to be well defined without regularisation. Hopefully, a method similar to the one in ref. [23] will work.

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APPENDIX A: SPINOR AND PURE SPINOR IDENTITIES IN $D = 11$

We will list some identities that have been useful for calculations.

Fierz rearrangements are always made between spinors at the right and left of two spinor products. The general Fierz identity reads

$$(AB)(CD) = \sum_{p=0}^5 \frac{1}{32p!} (C\gamma^{a_1 \dots a_p} B)(A\gamma_{a_p \dots a_1} D) \quad (A.1)$$

(with appropriate signs for statistics of operators). For bilinears in a pure spinor λ this reduces to

$$(A\lambda)(\lambda B) = -\frac{1}{64}(\lambda\gamma^{ab}\lambda)(A\gamma_{ab}B) + \frac{1}{3840}(\lambda\gamma^{abcde}\lambda)(A\gamma_{abcde}B) . \quad (A.2)$$

From the constraint on the spinor r , $(\bar{\lambda}\gamma^a r) = 0$, one derives

$$(\bar{\lambda}\gamma^{[ij}\bar{\lambda})(\bar{\lambda}\gamma^{kl]}r) = 0 . \quad (A.3)$$

Other useful relations among the non-minimal variables include

$$\begin{aligned} (\bar{\lambda}\gamma^i{}_k\bar{\lambda})(\bar{\lambda}\gamma^{jk}r) &= (\bar{\lambda}\gamma^{ij}\bar{\lambda})(\bar{\lambda}r) , \\ (\bar{\lambda}\gamma^i{}_k r)(\bar{\lambda}\gamma^{jk}r) &= (\bar{\lambda}\gamma^{ij}r)(\bar{\lambda}r) + \frac{1}{2}(\bar{\lambda}\gamma^{ij}\bar{\lambda})(rr) , \\ (\bar{\lambda}\gamma^i{}_k\bar{\lambda})(\bar{\lambda}\gamma^k{}_l r)(\bar{\lambda}\gamma^{lj}r) &= 0 , \\ (\bar{\lambda}\gamma^i{}_k r)(\bar{\lambda}\gamma^k{}_l r)(\bar{\lambda}\gamma^{lj}r) &= 0 . \end{aligned} \quad (A.4)$$

These can be used to show quite directly that $(\lambda\gamma_{ab}\lambda)[R_1^a, R_1^b] = 0$ (see Appendix B).

The symmetry of $(\bar{\lambda}\gamma^{a[b}\bar{\lambda})(\bar{\lambda}\gamma^{cd}r)(\bar{\lambda}\gamma^{e]f}r)$ is (af) , and no contraction is allowed, so this expression lies in the irreducible module (20010).

Various useful identities for a pure spinor λ include

$$\begin{aligned}
(\gamma_j \lambda)_\alpha (\lambda \gamma^{ij} \lambda) &= 0 , \\
(\gamma_i \lambda)_\alpha (\lambda \gamma^{abcdi} \lambda) &= 6(\gamma^{[ab} \lambda)_\alpha (\lambda \gamma^{cd]} \lambda) , \\
(\gamma_{ij} \lambda)_\alpha (\lambda \gamma^{abcij} \lambda) &= -18(\gamma^{[a} \lambda)_\alpha (\lambda \gamma^{bc]} \lambda) , \\
(\gamma_{ijk} \lambda)_\alpha (\lambda \gamma^{abijk} \lambda) &= -42\lambda_\alpha (\lambda \gamma^{ab} \lambda) , \\
(\gamma_{ij} \lambda)_\alpha (\lambda \gamma^{abcdij} \lambda) &= -24(\gamma^{[ab} \lambda)_\alpha (\lambda \gamma^{cd]} \lambda) , \\
(\gamma_i \lambda)_\alpha (\lambda \gamma^{abcdei} \lambda) &= \lambda_\alpha (\lambda \gamma^{abcde} \lambda) - 10(\gamma^{[abc} \lambda)_\alpha (\lambda \gamma^{de]} \lambda) ,
\end{aligned} \tag{A.5}$$

Alternative forms for R_2^a are:

$$\begin{aligned}
R_2^a &= -16\eta^{-3}(\bar{\lambda}\gamma^{a[b}\bar{\lambda})(\bar{\lambda}\gamma^{cd}r)(\bar{\lambda}\gamma^{e]f}r)(\lambda\gamma_{fb}\lambda)(\lambda\gamma_{cde}w) \\
&= \frac{4}{3}\eta^{-3}(\bar{\lambda}\gamma^{ab}\bar{\lambda})(\bar{\lambda}\gamma^{cd}r)(\bar{\lambda}\gamma^{ef}r)(\lambda\gamma_{bcde}{}^g\lambda)N_{fg} \\
&\quad - \frac{2}{3}\eta^{-3}(\bar{\lambda}\gamma^{ab}\bar{\lambda})(\bar{\lambda}\gamma^{cd}r)(\bar{\lambda}r)(\lambda\gamma_{bcd}{}^{ef}\lambda)N_{ef} \\
&= 2\eta^{-3}[\eta(\bar{\lambda}\gamma^{ab}r) - 2\phi(\bar{\lambda}\gamma^{ab}\bar{\lambda})](\bar{\lambda}\gamma^{cd}r)(\lambda\gamma_{bcd}w) \\
&= \{s, \eta^{-2}(\bar{\lambda}\gamma^{ab}\bar{\lambda})(\bar{\lambda}\gamma^{cd}r)\}(\lambda\gamma_{bcd}w) ,
\end{aligned} \tag{A.6}$$

where $N_{ab} = (\lambda\gamma_{ab}w)$ and $\phi = (\bar{\lambda}\gamma^{ij}r)(\lambda\gamma_{ij}\lambda)$.

APPENDIX B: CALCULATION OF A COMMUTATOR

In this Appendix, we will calculate the operator $R^a(\lambda\gamma_{ab}\lambda)R^b$ appearing in the master equation after the three-point coupling is introduced, and thus governing higher interactions. We can write $R^a(\lambda\gamma_{ab}\lambda)R^b = [R^a, (\lambda\gamma_{ab}\lambda)]R^b + \frac{1}{2}(\lambda\gamma_{ab}\lambda)[R^a, R^b]$.

Consider the first term. The only non-vanishing contribution comes from R_2^a . Using the form from Appendix A, $R_2^a = 2\eta^{-3}[\eta(\bar{\lambda}\gamma^{ab}r) - 2\phi(\bar{\lambda}\gamma^{ab}\bar{\lambda})](\bar{\lambda}\gamma^{cd}r)(\lambda\gamma_{bcd}w)$, one gets

$$[R^a, (\lambda\gamma^{bc}\lambda)] = 4\eta^{-3}[\eta(\bar{\lambda}\gamma^{ai}r) - 2\phi(\bar{\lambda}\gamma^{ai}\bar{\lambda})](\bar{\lambda}\gamma^{jk}r)(\lambda\gamma_{ijkbc}\lambda) . \tag{B.1}$$

If two of the indices are contracted, this gives zero thanks to $(\bar{\lambda}\gamma^{[ij}\bar{\lambda})(\bar{\lambda}\gamma^{kl]}r) = 0$.

The second term takes some more work. Examine first the terms in $[R^a, R^b]$ coming from the commutator of w with one of the prefactors η^{-k} . Using eq. (B.1) again, we get

$$[R^a, \eta] = -4[\eta(\bar{\lambda}\gamma^{ai}r) - 2\phi(\bar{\lambda}\gamma^{ai}\bar{\lambda})](\bar{\lambda}\gamma^{jk}r)(\lambda\gamma_{ijkbc}\lambda)(\bar{\lambda}\gamma^{bc}\bar{\lambda}) = 0 . \tag{B.2}$$

Now, the only remaining things to check are the terms from the anticommutator of the two D 's in R_1 and from the commutator of the w in R_2 with λ 's in R_1 and R_2 (except in η). Anticommuting the two D 's in R_1 gives

$$(\lambda\gamma_{ab}\lambda)[R_1^a, R_1^b] = \eta^{-3}(\bar{\lambda}\gamma^{ai}\bar{\lambda})(\bar{\lambda}\gamma^{bc}r)(\bar{\lambda}\gamma^{jk}r)(\lambda\gamma_{abc}\gamma^m\gamma_{ijk}\lambda)\partial_m . \quad (B.3)$$

Expanding the product of γ -matrices,

$$(\lambda\gamma_{abc}\gamma^m\gamma^{ijk}\lambda)\partial_m = 3(\lambda\gamma_{abc}^{[ij}\lambda)\partial^{k]} + 3(\lambda\gamma_{[ab}^{ijk}\lambda)\partial_{c]} - 9\delta_{[a}^i(\lambda\gamma_{bc]}^{jk]m}\lambda)\partial_m . \quad (B.4)$$

The corresponding three terms in eq. (B.3) vanish individually due to the identities (A.3) and (A.4) in Appendix A.

Similarly, the commutator between R_1 and R_2 gives

$$\begin{aligned} 2(\lambda\gamma_{ab}\lambda)[R_2^a, R_1^b] &= 4\eta^{-5}(\lambda\gamma_{ai}\lambda) [\eta(\bar{\lambda}\gamma^{ab}r) - 2\phi(\bar{\lambda}\gamma^{ab}\bar{\lambda})] (\bar{\lambda}\gamma^{cd}r) \\ &\quad \times (\bar{\lambda}\gamma^{ij}\bar{\lambda})(\bar{\lambda}\gamma^{kl}r)(\lambda\gamma_{bcd}\gamma_{jkl}D) . \end{aligned} \quad (B.5)$$

Here, we encounter the first non-vanishing contribution. Again using the relations from Appendix A, the second term in the square brackets vanishes. In the first term, there is the possibility to contract two indices between two separate pairs of matrices $(\lambda\gamma^{ab}\lambda)$ or $(\lambda\gamma^{ab}r)$, thus avoiding the zeroes of the last two identities in eq. (A.4). The result is

$$2(\lambda\gamma_{ab}\lambda)[R_2^a, R_1^b] = 24\eta^{-3}(\bar{\lambda}\gamma^{ab}\bar{\lambda})(\bar{\lambda}r)(rr)(\lambda\gamma_{ab}D) . \quad (B.6)$$

(From BRST invariance, it also necessary that the part with the smallest number of r 's does not contain $(\lambda\gamma^{(4)}D)$ or $(\lambda\gamma^{(6)}D)$.) Finally, if we write $R_1^a = M^{abcd}(\lambda\gamma_{bcd}D)$ and $R_2^a = \{s, M^{abcd}\}(\lambda\gamma_{bcd}w)$, the above result may be written

$$\begin{aligned} 2(\lambda\gamma_{ab}\lambda)[R_2^a, R_1^b] &= 2(\lambda\gamma_{ab}\lambda)\{s, M^{aijk}\}M^{blmn}(\lambda\gamma_{ijk}\gamma_{lmn}D) \\ &= [q, 24\eta^{-3}(\bar{\lambda}\gamma^{ab}\bar{\lambda})(\bar{\lambda}r)(rr)(\lambda\gamma_{ab}w)] , \\ (\lambda\gamma_{ab}\lambda)[R_2^a, R_2^b] &= 2(\lambda\gamma_{ab}\lambda)\{s, M^{aijk}\}\{s, M^{blmn}\}(\lambda\gamma_{ijk}\gamma_{lmn}w) \\ &= [s, 24\eta^{-3}(\bar{\lambda}\gamma^{ab}\bar{\lambda})(\bar{\lambda}r)(rr)(\lambda\gamma_{ab}w)] , \end{aligned} \quad (B.7)$$

and thus

$$(\lambda\gamma_{ab}\lambda)[R^a, R^b] = [Q, 24\eta^{-3}(\bar{\lambda}\gamma^{ab}\bar{\lambda})(\bar{\lambda}r)(rr)(\lambda\gamma_{ab}w)] . \quad (B.8)$$